

# Signal Analysis based on Complex Wavelet Signs

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## Abstract

We propose a new analysis tool for signals that is based on complex wavelet signs, called a signature. The complex-valued signature of a signal at some spatial location is defined as the fine-scale limit of the signs of its complex wavelet coefficients. We show that the signature equals zero at sufficiently regular points of a signal whereas at salient features, such as jumps or cusps, it is non-zero. At such feature points, the orientation of the signature in the complex plane can be interpreted as an indicator of local symmetry and antisymmetry. We establish that signature is invariant under fractional differentiation and rotates in the complex plane under fractional Hilbert transforms. Thus, the signature may be regarded as a complementary object to the local Sobolev regularity index. We derive an appropriate discretization, which shows that wavelet signatures can be computed explicitly. This allows an immediate application to signal analysis.

**Keywords:** Wavelet signature, Complex wavelets, Signal analysis, Hilbert transform, Fractional differentiation and integration, Phase, Sobolev regularity, Singular support, Randomized wavelet coefficients, Rademacher sequence, Salient feature

**2010 MSC:** 42C40, 94A12, 44A15

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## 1. Introduction

The determination and discrimination of salient features, such as jumps or cusps, is an important task in signal processing. Classical approaches assume the interesting features of a signal to be points of low regularity. In this context, local regularity is measured in terms of the (fractional) differentiability order, e.g., in the sense of local Hölder, Sobolev or Besov regularity. Since such measures of smoothness only rely on the *modulus* of wavelet coefficients, they do not take into account wavelet sign (or phase) information.

We may observe the shortcomings of a purely modulus based approach by considering the two functions  $f(x) = \operatorname{sgn} x$  and  $g(x) = 2 \log |x|$ . Since  $f$  and  $g$  are related by the Hilbert transform, their wavelet coefficients are equal with respect to the order of magnitude. Hence, the locally symmetric singularity of  $f$  and the locally antisymmetric singularity of  $g$  at the origin cannot be distinguished using a purely modulus-based signal analysis.

We present a new signal analysis tool, which is based on the *sign* of the wavelet coefficients. To this end, we investigate the fine scale limits of the signs of the wavelet coefficients

$$\sigma f(b) := \lim_{a \rightarrow 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle := \lim_{a \rightarrow 0} \frac{\langle f, \kappa_{a,b} \rangle}{|\langle f, \kappa_{a,b} \rangle|}, \quad (1)$$

where  $\kappa$  is a complex-valued wavelet,  $a > 0$  the scale, and  $b \in \mathbb{R}$  the location. The *complex-valued* quantity  $\sigma f(b)$  is called the *signature of  $f$  at location  $b$* . We shall see that the signature allows the local analysis

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of isolated salient features. Hereby, the orientation of the signature within the complex plane may be interpreted as an indicator of local symmetry or antisymmetry. In particular, we show that the signature is purely imaginary at a jump, whereas it is purely real at a cusp. Moreover, the signature is invariant under fractional Laplacians, i.e.,

$$\sigma((-\Delta)^{\frac{\alpha}{2}} f) = \sigma f,$$

and it serves as a multiplier when acting on the fractional Hilbert transform, in the sense that

$$\sigma(\mathcal{H}^\alpha f) = e^{i\alpha\frac{\pi}{2}} \sigma f.$$

Therefore, the signature may be interpreted as being “dual” to the local Sobolev regularity index, which is invariant under fractional Hilbert transforms but shifts under fractional Laplacians. We also establish that

$$\text{sing supp } f \not\subset \text{supp } \sigma f \quad \text{and} \quad \text{supp } \sigma f \not\subset \text{sing supp } f.$$

Thus, a singularity in the classical sense need not coincide with a signature-type singularity. Finally, the signature classifies each point of a signal, whose signs are randomly disturbed, as a non-salient point.

We also propose a method to numerically compute the signature of real world signals and validate the theoretically developed concepts by numerical experiments.

### 1.1. Related approaches and our contribution

The local decay rate of the modulus of wavelet coefficients is often used to characterize the singular support of a function. It even provides a means of extracting the local regularity. (See, for instance, [1].) The local decay rate is also used in the context of detection and classification of salient features in signals and images. Attempts along these lines can be found in the book by Mallat, see [2, Chapter 6.2-6.3] and references therein, where the procedure is called wavelet transform modulus maxima and applied to multiscale edge detection. However, modulus based methods do not, and can not, take into account the information contained in the sign (or phase) of the wavelet coefficients.

This stands in stark contrast to the importance of sign information in the signal structure. For instance, it was shown in [3] that bandpass signals are characterized, up to a constant, by their zero crossings, which in turn are determined by the changes in the sign of the function values. Thus, when dealing with bandpass wavelets, the wavelet coefficients are determined by their sign. In this context, the well known experiments by Oppenheim and Lim [4] indicate that sign information is more important for the reconstruction of images than the modulus.

The sign information of the Fourier coefficients is used in signal analysis via *phase congruency* [5, 6]. In [7], Kovessy added a multiscale aspect to phase congruency and applied it successfully to signal analysis. However, the approach was not derived rigorously and involves several heuristic methods and principles. Regarding the signs of complex wavelets, it has been observed in [8] and [9] that the lines of constant phase in the continuous wavelet transform of a Gaussian complex wavelet converge towards the singularities in the signal. However, to the authors’ knowledge, these observations have not been cast into rigorous statements.

We introduce a new approach to incorporate the sign of complex wavelet coefficients into signal analysis. In particular, we use the complex-valued *signature* to delineate singularities in signals. This novel method is based on *complex wavelets*, which are complex-valued functions whose real and imaginary parts are Hilbert transforms of each other. Complex wavelets have been successfully applied in various signal and image processing applications, for instance in denoising [10, 11]. In this paper, we use the class of complex bandpass wavelets that have a one-sided and positive Fourier spectrum (Definition 1). This class includes, for instance, the complex Meyer-type wavelets. The main idea behind the signature of a signal  $f$  is to investigate the fine scale behavior of the complex wavelet signs,  $\text{sgn} \langle f, \kappa_{a,b} \rangle$ , defined as

$$\text{sgn} \langle f, \kappa_{a,b} \rangle = \frac{\langle f, \kappa_{a,b} \rangle}{|\langle f, \kappa_{a,b} \rangle|},$$

if  $\langle f, \kappa_{a,b} \rangle \neq 0$  and zero otherwise. The signature  $\sigma f$  at a location  $b \in \mathbb{R}$  is equal to the limit

$$\sigma f(b) = \lim_{a \rightarrow 0} \text{sgn} \langle f, \kappa_{a,b} \rangle,$$

if the limit exists *for all* wavelets  $\kappa$  in our class and if the limit is independent of the choice of  $\kappa$ ; otherwise, we set  $\sigma f(b) = 0$ ; see [Definition 2](#). We consider  $b \in \mathbb{R}$  to be a salient point of a signal  $f$  if the signature at  $b$  is not equal to 0. Indeed, if  $f$  has a jump discontinuity at  $b$ , we prove in [Theorem 12](#) that the limit equals  $\pm i$ , whereas if  $f$  is of cusp-type at  $b$ , we observe in [Example 9](#) that the limit equals  $\pm 1$ . Furthermore, if the signal is locally polynomial around  $b$ , i.e. regular, then either the above limit does not exist, or equals zero. ([Theorem 5](#) and [Corollary 6](#)) These results show that signature is capable of distinguishing between different types of singularities and may be used to classify them. Hereby, the orientation of the signature within the complex plane can be seen as an indicator of local symmetry or antisymmetry. This coincides with the analogous interpretations for phase congruency [[5](#), [6](#), [7](#)]. A similar connection between the fine scale wavelet phase and the local geometry of a singularity was observed in [[12](#), Chapter 4.3].

We set the signature into relation to the classical local smoothness analysis in terms of Sobolev regularity. The local Sobolev regularity index of a function is invariant under the application of the fractional Hilbert transform but changes under the action of the fractional Laplacian [[13](#)]. We show that the signature is “dual” to the local Sobolev regularity index in the sense that the fractional Hilbert transform acts like a multiplier on  $\sigma f$ , whereas  $\sigma f$  remains invariant under the application of the fractional Laplacian; see [Theorem 14](#) and [Theorem 15](#).

In [[14](#)] it is shown that random modifications of the wavelet signs significantly alter the shape of a signal. Since the moduli of the wavelet coefficients remain unchanged so does the classical modulus based regularity. In [Theorem 19](#) we show that such a random modification of the signs of the wavelet coefficients yields a signature identically equal to zero. Hence, a signal of randomly disturbed wavelet signs does not possess any salient features with respect to our definition of signature.

In order to compute the signature of a real world signal  $f$ , we introduce a discretization based on the mean values

$$\bar{w}_b := \frac{1}{N} \sum_{j=1}^N \operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle = \frac{1}{N} \sum_{j=1}^N \frac{\langle f, \kappa_{a_j, b} \rangle}{|\langle f, \kappa_{a_j, b} \rangle|}. \quad (2)$$

where  $\{a_j\}_{j=1}^\infty$  is a discrete zero sequence of scale samples. The limit  $N \rightarrow \infty$  of (2) is the Cesàro limit of the sequence of signs  $\{\operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle\}_{j \in \mathbb{N}}$ , which converges to  $\sigma f(b)$ , if  $\sigma f(b) \neq 0$ . This motivates the definition of discrete signature  $\bar{\sigma} f$  on a finite set of scale samples  $\{a_j\}_{j=1}^N$ :

$$\bar{\sigma} f(b) := \begin{cases} \operatorname{sgn} \bar{w}_b, & \text{if } |\bar{w}_b| > \tau, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where  $\tau \in ]0, 1[$  is an empirical threshold parameter. Interestingly, a directional statistics approach to a sign based signal analysis leads to the same formula (3). This statistical derivation of the discrete signature is developed in the thesis of the third author [[15](#)].

Our discrete signature (3) has some relations to the phase congruency of [[7](#)]. In one dimension and adapted to our notation, phase congruency PC is defined in [[7](#)] as the modulus of the complex quantity

$$\widetilde{\text{PC}}(b) = \frac{\sum_{j=1}^N \langle f, \kappa_{a_j, b} \rangle}{\sum_{j=1}^N |\langle f, \kappa_{a_j, b} \rangle|}.$$

Note that the normalization is carried out *after* summation, in contrast to our definition of discrete signature (2) and (3). A major issue in phase congruency is the presence of a dominating coefficient yielding a large phase congruency  $\text{PC}(b) = |\widetilde{\text{PC}}(b)|$ , even though the sign values of the coefficient may point into arbitrary directions [[7](#)]. In [[7](#)], this issue was resolved heuristically at the cost of introducing two additional empirical parameters. In our discretization (3), we do not encounter this problem since every coefficient contributes the same weight, no matter how small or large the modulus is.

## 1.2. Organization of this article

The structure of this article is as follows. In [Section 2](#) we introduce the new concept of signature and present some of its basic properties. We then compute the signature of some specific signals and investigate

the behavior of signature under the fractional Laplacian and the fractional Hilbert transform. Finally, we set up relations to moduli based signal analyses and to the randomization of wavelet coefficients. In order to compute the signature for real world problems, we introduce in [Section 3](#) a discretization of signature and illustrate our theoretical results by numerical experiments.

## 2. Signal analysis by signature

In this section, we introduce the new concept of complex signature for one-dimensional signals. In addition we discuss important properties of the complex signature and present various examples. Furthermore, we discuss the behavior of signature under operations such as the fractional Laplacian and the fractional Hilbert transform and contrast it with the behavior of the classical Sobolev regularity index. We close the section by showing that signals, whose wavelet signs are randomly disturbed, have signature zero at every location.

### 2.1. Definitions and basic properties

We define the Fourier transform of a Schwartz function  $f \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  by

$$\mathcal{F}(f)(\omega) := \widehat{f}(\omega) := \int_{\mathbb{R}} e^{-i\omega x} f(x) dx.$$

Likewise, we use the above notation for the usual extension to the space of tempered distributions  $\mathcal{S}'(\mathbb{R}; \mathbb{C})$ . Furthermore,  $\mathcal{F}^{-1}(f)$  and  $f^\vee$  denote the corresponding inverse Fourier transform of  $f$ .

Let us introduce the class of complex wavelets we need for the definition of signature.

**Definition 1.** *We call a complex-valued non-zero function  $\kappa \in \mathcal{S}(\mathbb{R}; \mathbb{C})$  a signature wavelet if  $\kappa$  has a one-sided compactly supported Fourier transform, i.e.,*

$$\text{supp } \widehat{\kappa} \subseteq [c, d], \quad 0 < c < d < \infty, \quad (4)$$

*and a non-negative frequency spectrum, i.e.,*

$$\widehat{\kappa}(\omega) \geq 0, \quad \text{for all } \omega \in \mathbb{R}. \quad (5)$$

The wavelet system associated with a signature wavelet  $\kappa$  is defined as the family of functions

$$\kappa_{a,b}(x) := \frac{1}{\sqrt{a}} \kappa\left(\frac{x-b}{a}\right), \quad \text{where } a > 0 \text{ and } b \in \mathbb{R}.$$

An example of a signature wavelet is given by the inverse Fourier transform of the (one-sided) Meyer window  $W$ , i.e.,

$$\kappa(x) = \mathcal{F}^{-1}(W)(x). \quad (6)$$

We refer to [Appendix A](#) for the exact definition of the Meyer window function  $W$ . The graph of  $\kappa$  is depicted in [Figure 1](#). We employ this signature wavelet in the numerical experiments in [Section 3](#).

Recall that the sign of a complex number  $z \in \mathbb{C}$  is defined by

$$\text{sgn } z = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

The signature of a signal is then defined as follows.

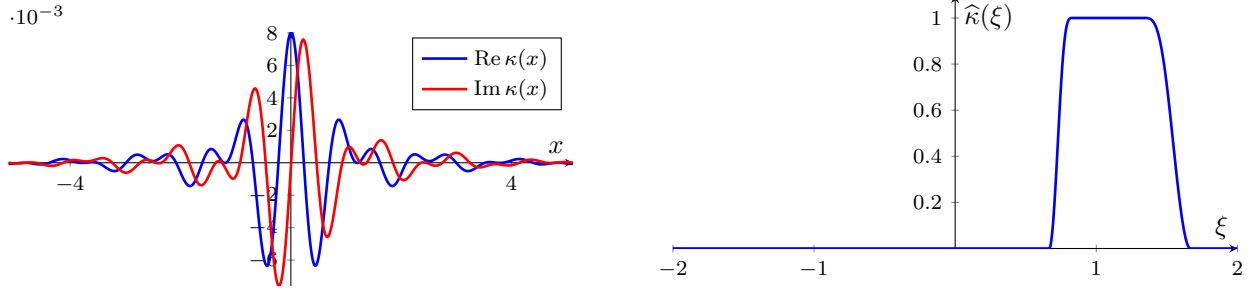


Figure 1: The Meyer-type signature wavelet  $\kappa$  (left) and its Fourier transform  $\hat{\kappa}$  (right).

**Definition 2.** Let  $f \in \mathcal{S}'(\mathbb{R}; \mathbb{R})$ . If there exists  $z \in \mathbb{C}$ , such that for all signature wavelets  $\kappa$ ,

$$\lim_{a \rightarrow 0} \operatorname{sgn} \langle f, \kappa_{a,b} \rangle = z,$$

then we define the signature,  $\sigma f$ , of  $f$  at  $b \in \mathbb{R}$  by

$$\sigma f(b) := z;$$

otherwise, we set

$$\sigma f(b) := 0.$$

Note that the signature  $\sigma f(b)$  is either equal to zero or is a complex number of modulus 1. It follows directly from the definition that the signature is invariant under translations, i.e.,

$$\sigma(T_r f)(b) = (\sigma f)(b - r) \quad (7)$$

and under dilations, i.e.,

$$\sigma(D_\nu f)(b) = (\sigma f)(\nu b). \quad (8)$$

Here, the operator of translation by  $r \in \mathbb{R}$ ,  $T_r$ , and dilation by  $\nu \in \mathbb{R} \setminus \{0\}$ ,  $D_\nu$ , is defined by

$$T_r f(x) := f(x - r) \quad \text{and} \quad D_\nu f(x) := \frac{1}{\sqrt{|\nu|}} f\left(\frac{x}{\nu}\right),$$

respectively.

Since the Fourier transform of a signature wavelet  $\kappa$  vanishes in a neighborhood of the origin, we have that

$$\langle p, \kappa \rangle = 0, \quad \text{for any polynomial } p. \quad (9)$$

Therefore, the signature is well defined on the tempered distributions modulo polynomials  $\mathcal{S}'/\mathcal{P}$ , where  $\mathcal{P}$  denotes the space of all polynomials.

## 2.2. Signatures of specific signals

In this section, we compute the signature of specific parts of a signal. We prove that the signature  $\sigma f(b)$  is equal to zero if  $f$  coincides with a polynomial locally around  $b$ . Furthermore, we show that the signature equals  $\pm i$  at jumps to the right or to the left. Pure cusps yield an signature of  $\pm 1$ , depending on whether the cusp points upwards or downwards.

Let us begin by considering some elementary examples for which the value of the signature follows directly from the definition.

**Example 3.** 1. Consider a polynomial  $p$  and any signature wavelet  $\kappa$ . The vanishing of all moments of  $\kappa$  implies

$$\langle p, \kappa_{a,b} \rangle = 0, \quad \text{for all } a > 0 \text{ and } b \in \mathbb{R}.$$

Hence, polynomials have signature equal to zero on the entire real line. Note that this assertion is independent of the choice of  $\kappa$ .

2. Let us compute the signature for the cosine function:

$$\langle \cos, \kappa_{a,b} \rangle = \langle \widehat{\cos}, (\kappa_{a,b})^\vee \rangle = \pi \langle \delta_1 + \delta_{-1}, (\kappa_{a,b})^\vee \rangle = \pi \sqrt{a} e^{-ib} \widehat{\kappa}(a). \quad (10)$$

Since  $\widehat{\kappa}(a) = 0$  for sufficiently small  $a$ , we obtain

$$\sigma(\cos)(b) = 0,$$

for any  $b \in \mathbb{R}$ .

3. A similar computation applies to the sine function, and thus to all trigonometric polynomials, yielding the same result.
4. In general, since the Fourier transform of a signature wavelet  $\kappa$  vanishes in a neighborhood of the origin, we obtain for any band-limited tempered distribution  $f$  and for each  $b \in \mathbb{R}$ ,

$$\operatorname{sgn} \langle f, \kappa_{a,b} \rangle = \operatorname{sgn} \langle f^\vee, \widehat{\kappa_{a,b}} \rangle = 0, \quad \text{for sufficiently small } a.$$

Hence, the signature of a band-limited tempered distribution is equal to zero for every  $b \in \mathbb{R}$ .

To establish that the signature is equal to zero at sufficiently regular points of a signal ([Theorem 5](#) and [Corollary 6](#)), we require the following lemma about the zeros of functions all of whose moments vanish.

**Lemma 4.** *Let  $g$  be a real-valued continuous function of rapid decay such that*

$$\int_{\mathbb{R}} g(x) x^k dx = 0, \quad \text{for all } k \in \mathbb{N}_0. \quad (11)$$

*If the support of  $g$  is not compact, then there exists an unbounded monotone sequence  $\{x_n\}_{n \in \mathbb{N}}$  of sign changes of  $g$ , i.e., there exists  $\{x_n\}_{n \in \mathbb{N}}$  such that  $|x_n| \rightarrow \infty$ ,  $\operatorname{sgn} g(x_{2n}) = 1$  and  $\operatorname{sgn} g(x_{2n+1}) = -1$ . In particular, the set of zeros  $\{x \in \mathbb{R} : g(x) = 0\}$  is unbounded.*

*Proof.* Assume to the contrary that there exists no such sequence. Then we can find an  $a > 0$  such that  $g$  neither changes sign on  $] -\infty, -a[$  nor on  $]a, \infty[$ . The continuity of  $g$  implies that

$$\left| \int_{-a}^a \left( \frac{x}{a} \right)^m g(x) dx \right| \leq 2a \max_{x \in [-a, a]} |g(x)| =: C, \quad \forall m \in \mathbb{N}_0, \quad (12)$$

where  $C \geq 0$ .

Since  $g$  is continuous and not compactly supported, we find an interval  $[c, d]$  in  $\mathbb{R} \setminus [-a, a]$  and a constant  $K > 0$  such that  $|g| > K$  on  $[c, d]$ . Suppose then, without loss of generality, that  $[c, d] \subset ]a, \infty[$  and that  $g > K$  on  $[c, d]$ . Then there exists an  $m_0 \in \mathbb{N}$  such that for  $m \geq m_0$

$$\int_{[c, d]} \left( \frac{x}{a} \right)^m g(x) dx > C.$$

If  $g \geq 0$  on  $] -\infty, -a[$ , then choose an even  $m \geq m_0$ , and if  $g \leq 0$  on  $] -\infty, -a[$ , then choose an odd  $m \geq m_0$ . In either case,

$$\int_{\mathbb{R} \setminus [-a, a]} \left( \frac{x}{a} \right)^m g(x) dx > C,$$

which, together with (12), implies that

$$\int_{\mathbb{R}} \left(\frac{x}{a}\right)^m g(x) dx > 0,$$

contradicting the vanishing moment condition (11).  $\square$

In the following, we say that a function  $f$  is of *polynomial growth* if there exists an  $N \in \mathbb{N}$  such that  $f \in O(|x|^N)$  as  $|x| \rightarrow \infty$ . Next, we show that a signal of polynomial growth has signature equal to zero at a point where all derivatives are equal to zero.

**Theorem 5.** *Let  $f$  be a real-valued, locally integrable function of polynomial growth. Further assume that  $f$  is smooth in a neighborhood of  $b \in \mathbb{R}$ . If*

$$f^{(k)}(b) = 0, \quad \text{for all } k \in \mathbb{N}_0,$$

then

$$\sigma f(b) = 0.$$

In particular,

$$\text{supp } \sigma f \subseteq \text{supp } f.$$

*Proof.* As signature is invariant under translations (cf. (7)), it suffices to prove the claim for  $b = 0$ . Since  $f$  has polynomial growth, there exists an  $N \in \mathbb{N}$  such that  $f \in O(x^N)$  as  $|x| \rightarrow \infty$ . We define a function

$$H(u) = u^{N+1} \int_{\mathbb{R}} f(x) \kappa(ux) dx, \quad (13)$$

and show that  $H$  has infinitely many vanishing moments. To this end, we consider the following integrals:

$$\int_{\mathbb{R}} H(u) u^k du = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \kappa(ux) dx u^{k+N+1} du = \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(ux) u^{k+N+1} du f(x) dx, \quad k \in \mathbb{N}_0. \quad (14)$$

We need to justify the interchange of the order of integration. By a change of variables  $u = \frac{y}{x}$ , for  $x \neq 0$ , we get for the absolute value of the inner integral the following estimate:

$$\int_{\mathbb{R}} |\kappa(ux)| |u|^{k+N+1} du = \frac{1}{|x|^{k+N+2}} \int_{\mathbb{R}} |\kappa(y)| |y|^{k+N+1} dy \leq \frac{C_k}{|x|^{k+N+2}},$$

where  $C_k$  is a constant depending only on  $k$ . Therefore,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\kappa(ux)| \cdot |u|^{k+N+1} du |f(x)| dx \leq C_k \int_{\mathbb{R}} \frac{|f(x)|}{|x|^{k+N+2}} dx < \infty, \quad \text{for all } k \in \mathbb{N}_0.$$

To justify the last inequality in (14), let  $\epsilon > 0$  and  $\delta > \epsilon$  sufficiently large. Then, since  $f^{(l)}(0) = 0$  for all  $l \in \mathbb{N}_0$ ,

$$\int_{-\epsilon}^{\epsilon} \frac{|f(x)|}{|x|^{k+N+2}} dx < \infty.$$

Moreover, since  $f$  is locally integrable,

$$\left( \int_{-\delta}^{-\epsilon} + \int_{\epsilon}^{\delta} \right) \frac{|f(x)|}{|x|^{k+N+2}} dx < \infty.$$

Furthermore, as  $f \in O(x^N)$ ,

$$\left( \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{|f(x)|}{|x|^{k+N+2}} dx < \infty.$$

Hence, the right hand side of (14) exists in absolute value and thus the interchange of the order of integration in (14) is justified by the Fubini-Tonelli theorem.

Now, since all moments of the signature wavelet  $\kappa$  vanish, we get for  $x \neq 0$  that

$$\int_{\mathbb{R}} \kappa(ux) u^{k+N+1} du = 0.$$

Substitution into (14) yields

$$\int_{\mathbb{R}} H(u) u^k du = \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(ux) u^{k+N+1} du f(x) dx = 0, \quad \text{for all } k \in \mathbb{N}_0. \quad (15)$$

We furthermore observe that  $H$  is a continuous function of rapid decay. Thus  $\operatorname{Re} H$  and  $\operatorname{Im} H$  are continuous functions of rapid decay all of whose moments vanish. First assume that  $H$  is not compactly supported. Then, by Lemma 4, we find an unbounded monotone sequence  $\{u_n\}_{n \in \mathbb{N}}$  of sign changes of  $\operatorname{Re} H$ , that is,

$$\operatorname{sgn} \operatorname{Re} H(u_n) = \begin{cases} +1, & n \text{ even;} \\ -1, & n \text{ odd.} \end{cases}$$

Analogously, there exists an unbounded monotone sequence  $\{v_n\}_{n \in \mathbb{N}}$  of sign changes of  $\operatorname{Im} H$ . Since  $\kappa$  is a Hermitian function and  $f$  is real-valued,  $H$  is also Hermitian, i.e.,

$$H(-u) = \overline{H(u)}.$$

By this symmetry property, we may assume that both sequences  $\{u_n\}$  and  $\{v_n\}$  are increasing and tend to  $+\infty$ . Thus, the limit

$$\lim_{u \rightarrow \infty} \operatorname{sgn} H(u)$$

cannot exist. If  $H$  is compactly supported, then the limit equals zero.

To conclude the proof, we observe that for  $u > 0$

$$\operatorname{sgn} \left\langle f, \kappa_{\frac{1}{u}, 0} \right\rangle = \operatorname{sgn} \left( |u|^{\frac{1}{2}} \int f(x) \kappa(ux) dx \right) = \operatorname{sgn} H(u).$$

Thus, with  $a = \frac{1}{u}$ , the limit

$$\lim_{a \rightarrow 0+} \operatorname{sgn} \langle f, \kappa_{a, 0} \rangle = \lim_{u \rightarrow \infty} \operatorname{sgn} \left\langle f, \kappa_{\frac{1}{u}, 0} \right\rangle$$

either does not exist or equals zero. In either case,  $\sigma f(0) = 0$ , which completes the proof.  $\square$

A particularly interesting consequence of Theorem 5 is the following result.

**Corollary 6.** *Let  $f$  be a real-valued, locally integrable function of polynomial growth which is smooth in a neighborhood of  $b \in \mathbb{R}$ . If for some  $k_0 \in \mathbb{N}_0$ ,  $f^{(k)}(b) = 0$ , for all  $k \geq k_0$ , then*

$$\sigma f(b) = 0.$$

*In particular, if  $f$  coincides on an open set  $U \subset \mathbb{R}$  with a polynomial then  $\sigma f(b) = 0$ , for every  $b \in U$ .*

*Proof.* Let  $p$  be a polynomial of order  $k_0 - 1$  such that  $p^{(k)}(b) = f^{(k)}(b)$  for all  $k \leq k_0 - 1$ . Hence,  $f - p$  satisfies the hypotheses of Theorem 5 implying that  $\sigma(f - p)(b) = 0$ . Since  $\langle f - p, \kappa_{a, b} \rangle = \langle f, \kappa_{a, b} \rangle$ , we obtain  $\sigma(f - p)(b) = \sigma f(b) = 0$ . The second claim is a direct consequence of the first.  $\square$



In the next example, we consider the unit step function. Here, we can compute the signature at  $b = 0$  explicitly. For  $b \neq 0$ , we can apply [Corollary 6](#).

**Example 7.** Let  $U$  be the unit step (or Heaviside) function defined by

$$U(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{else.} \end{cases}$$

For any signature wavelet  $\kappa$ , we have that

$$\langle U, \kappa_{a,0} \rangle = \langle \widehat{U}, (\kappa_{a,0})^\vee \rangle = \sqrt{a} \int_{\mathbb{R}} \frac{\kappa^\vee(a\xi)}{i\pi\xi} d\xi = \frac{i\sqrt{a}}{\pi} \int_{\mathbb{R}} \frac{\widehat{\kappa}(a\xi)}{\xi} d\xi. \quad (16)$$

Hence, since  $\widehat{\kappa} \geq 0$  and  $\text{supp } \widehat{\kappa} \subset [0, \infty)$ , we obtain that  $\text{sgn } \langle U, \kappa_{a,0} \rangle = i$ , for all  $a > 0$ . For  $b \neq 0$ , we apply [Corollary 6](#) yielding

$$\sigma U(b) = \begin{cases} i, & \text{if } b = 0, \\ 0, & \text{else.} \end{cases}$$

Note that the same arguments can be applied to the sign function  $x \mapsto 2U(x) - 1$ .

Next we show that the signature of a homogeneous function vanishes away from the origin.

**Theorem 8.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a homogeneous function of degree  $\gamma \in \mathbb{R}$ . Then,

$$\sigma f(b) = 0, \quad \text{for any } b \neq 0. \quad (17)$$

*Proof.* Let  $\kappa$  be a signature wavelet and let  $b \neq 0$  be fixed. For any  $a > 0$ , we have that

$$\langle f, \kappa_{a,b} \rangle = \langle f^\vee, \widehat{\kappa_{a,b}} \rangle = \sqrt{a} \int_{\mathbb{R}} f^\vee(\omega) \widehat{\kappa}(a\omega) e^{-ib\omega} d\omega = a^{-\frac{1}{2}} \int_{\mathbb{R}} f^\vee(a^{-1}\xi) \widehat{\kappa}(\xi) e^{-ib\xi/a} d\xi.$$

Since  $f$  is homogeneous of degree  $\gamma$ ,  $f^\vee$  is homogeneous of degree  $-\gamma - 1$ . (See [\[16, Theorem 7.1.16\]](#).) Hence,

$$\langle f, \kappa_{a,b} \rangle = a^{\gamma-\frac{1}{2}} \int_{\mathbb{R}} f^\vee(\xi) \widehat{\kappa}(\xi) e^{-ib\xi/a} d\xi.$$

Setting  $u = \frac{1}{a}$ , we obtain

$$\langle f, \kappa_{1/u,b} \rangle = u^{-\gamma+\frac{1}{2}} \int_{\mathbb{R}} f^\vee(\xi) \widehat{\kappa}(\xi) e^{-ib\xi u} d\xi = u^{-\gamma+\frac{1}{2}} \mathcal{F}(f^\vee \cdot \widehat{\kappa})(bu). \quad (18)$$

The function  $\xi \mapsto \widehat{\kappa}(\xi) \cdot f^\vee(\xi)$  is in  $\mathcal{C}^\infty$  and vanishes on a neighborhood of the origin. Arguments analogous to those in the proof of [Theorem 5](#) imply that the function  $u \mapsto \mathcal{F}(f^\vee \cdot \widehat{\kappa})(bu)$  is of rapid decay and has infinitely many vanishing moments. Finally, by [Lemma 4](#), we obtain that  $\mathcal{F}(f^\vee \cdot \widehat{\kappa})(bu)$  has infinitely many zeros. Hence,

$$\text{sgn } \langle f, \kappa_{1/u,b} \rangle = \text{sgn } \left( u^{-\gamma+\frac{1}{2}} \mathcal{F}(f^\vee \cdot \widehat{\kappa})(bu) \right)$$

cannot converge for  $u \rightarrow \infty$  to a non-zero value. Therefore,  $\sigma f(b) = 0$ .  $\square$

Using [Theorem 8](#), we may compute the signature at pure cusp-type singularities.

**Example 9.** Consider the function

$$f(x) = |x - x_0|^\gamma, \quad \text{where } \gamma > 0.$$

For  $0 < \gamma < 1$ , this function resembles a pure cusp singularity. By translation invariance it is sufficient to compute the signature of  $x \mapsto |x|^\gamma$ . This function requires careful differentiation with respect to  $\gamma$ . If  $\gamma$  is an even integer, we are in the case of polynomials and the signature is identically equal to zero. (Compare with (9).) If  $\gamma$  is an odd integer then, according to [17, eq. 2.3(19)], we obtain

$$\langle |\bullet|^\gamma, \kappa_{a,0} \rangle = \left\langle \widehat{|\bullet|^\gamma}, (\kappa_{a,0})^\vee \right\rangle = 2(-1)^{\frac{\gamma-1}{2}+1} \gamma! \int_{\mathbb{R}} \frac{(\kappa_{a,0})^\vee(\xi)}{|\xi|^{\gamma+1}} d\xi. \quad (19)$$

For  $\gamma > 0$  and  $\gamma \notin \mathbb{N}$ , it follows from [17, eq. 2.3(12)] that

$$\langle |\bullet|^\gamma, \kappa_{a,0} \rangle = \left\langle \widehat{|\bullet|^\gamma}, (\kappa_{a,0})^\vee \right\rangle = -\sin\left(\frac{\pi\gamma}{2}\right) \Gamma(\gamma+1) \int_{\mathbb{R}} \frac{(\kappa_{a,0})^\vee(\xi)}{|\xi|^{\gamma+1}} d\xi. \quad (20)$$

Therefore, the wavelet signs at  $b = 0$  are given by

$$\text{sgn } \langle |\bullet|^\gamma, \kappa_{a,0} \rangle = \begin{cases} 0, & \text{if } \gamma \in 2\mathbb{N}_0, \\ -1, & \text{if } \gamma \in ]0, 2[ \cup ]4, 6[ \cup \dots, \\ +1, & \text{if } \gamma \in ]2, 4[ \cup ]6, 8[ \cup \dots, \end{cases}$$

for every  $a > 0$ . Now, Theorem 8 implies

$$\sigma(|\bullet|^\gamma)(b) = 0, \quad \text{for all } b \neq 0.$$

Hence, by translation invariance, it follows that

$$\sigma f(x_0) = \begin{cases} 0, & \text{if } \gamma \in 2\mathbb{N}_0, \\ -1, & \text{if } \gamma \in ]0, 2[ \cup ]4, 6[ \cup \dots, \\ +1, & \text{if } \gamma \in ]2, 4[ \cup ]6, 8[ \cup \dots \end{cases}$$

and

$$\sigma f(b) = 0, \quad \text{for } b \neq x_0.$$

Having computed the signature of some basic signals, we now consider the superposition of two signals.

**Proposition 10.** *Let  $b \in \mathbb{R}$  and let  $f, g \in \mathcal{S}'(\mathbb{R}, \mathbb{R})$  be two tempered distributions such that  $\sigma f(b) \neq 0$  and*

$$\langle g, \kappa_{a,b} \rangle \in o(\langle f, \kappa_{a,b} \rangle),$$

*for all signature wavelets  $\kappa$ . (Here, the Landau symbol  $o$  refers to  $a \rightarrow 0$ .) Then*

$$\sigma(f+g)(b) = \sigma f(b).$$

*Proof.* Let  $\kappa$  be a signature wavelet. From  $\sigma f(b) \neq 0$ , it follows that  $\langle f, \kappa_{a,b} \rangle \neq 0$  for sufficiently small  $a$ . Therefore,

$$\frac{\langle f+g, \kappa_{a,b} \rangle}{|\langle f+g, \kappa_{a,b} \rangle|} = \frac{\langle f, \kappa_{a,b} \rangle \left(1 + \frac{\langle g, \kappa_{a,b} \rangle}{\langle f, \kappa_{a,b} \rangle}\right)}{\left|\langle f, \kappa_{a,b} \rangle \left(1 + \frac{\langle g, \kappa_{a,b} \rangle}{\langle f, \kappa_{a,b} \rangle}\right)\right|} \xrightarrow{a \rightarrow 0} \sigma f(b),$$

which completes the proof.  $\square$

The following example, which considers the superposition of the unit step function and a pure cusp, illustrates Proposition 10.

**Example 11.** Let  $f$  be the superposition of a cusp and the Heaviside function, i.e.,

$$f(x) = U(x) + |x|^\gamma, \quad 0 < \gamma < 1.$$

From (19) and (20), we obtain

$$|\langle |\bullet|^\gamma, \kappa_{a,0} \rangle| \sim |a|^{\gamma+1/2},$$

and from (16), we get

$$|\langle U, \kappa_{a,0} \rangle| \sim |a|^{1/2}.$$

This shows that

$$\langle |\bullet|^\gamma, \kappa_{a,0} \rangle \in o(\langle U, \kappa_{a,0} \rangle),$$

for all signature wavelets  $\kappa$ . Hence, we can apply Proposition 10 and conclude that

$$\sigma f(0) = \sigma U(0) = i. \quad (21)$$

Note that the less regular step singularity dominates over the cusp singularity.

Next we show that, in general, a jump discontinuity induces a purely imaginary signature at the jump location. A function  $f$  has a *jump (or step) discontinuity* at  $b$  if the left-hand and the right-hand limits  $f(b+)$  and  $f(b-)$  exist but are not equal.

**Theorem 12.** Let  $f$  be a real-valued, locally integrable function of polynomial growth and let  $b \in \mathbb{R}$ . If there exists a neighborhood  $U$  of  $b$  such that  $f$  is continuous on  $U \setminus \{b\}$  and has a jump discontinuity at  $b$ , then

$$\sigma f(b) = \begin{cases} +i, & \text{if } f(b-) < f(b+), \\ -i, & \text{if } f(b-) > f(b+). \end{cases} \quad (22)$$

*Proof.* By the translation invariance of the signature (7), it is sufficient to prove the result for  $b = 0$ . First assume that the function  $f$  jumps upwards from the left to the right, that is,  $\Theta := f(b+) - f(b-) > 0$ .

Throughout this proof we write  $f^S$  for the symmetric part and  $f^A$  for the antisymmetric part of  $f$ , that is,

$$f^S(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f^A(x) = \frac{f(x) - f(-x)}{2}.$$

There exists a continuous extension of  $f^S$  around the origin, which we denote again by  $f^S$ , so that  $\lim_{x \rightarrow 0+} f^S(x) = \lim_{x \rightarrow 0-} f^S(x) = f^S(0)$ . By the vanishing moments property of signature wavelets, we may assume that  $f^S(0) = 0$ .

Let  $\kappa$  be a signature wavelet. Since  $\widehat{\kappa}$  is real-valued,  $\text{Re } \kappa_{a,0}$  is symmetric and  $\text{Im } \kappa_{a,0}$  is antisymmetric. Thus we have

$$\text{Re } \langle f, \kappa_{a,0} \rangle = \langle f^S + f^A, \text{Re } \kappa_{a,0} \rangle = \langle f^S, \text{Re } \kappa_{a,0} \rangle$$

and

$$\text{Im } \langle f, \kappa_{a,0} \rangle = \langle f^S + f^A, \text{Im } \kappa_{a,0} \rangle = \langle f^A, \text{Im } \kappa_{a,0} \rangle.$$

We observe that

$$C_\kappa := \int_0^\infty \text{Im } \kappa(x) dx = \text{Im } \int_{\mathbb{R}} U(x) \kappa(x) dx = \text{Im } \int_{\mathbb{R}} -\frac{1}{i\omega} \widehat{\kappa}(\omega) d\omega = \int_{\mathbb{R}} \frac{1}{\omega} \widehat{\kappa}(\omega) d\omega > 0.$$

We show that

$$\frac{1}{\sqrt{a}} \langle f^S, \text{Re } \kappa_{a,0} \rangle = \frac{1}{a} \int_{\mathbb{R}} f^S(x) \text{Re } \kappa\left(\frac{x}{a}\right) dx \rightarrow 0, \quad \text{as } a \rightarrow 0, \quad (23)$$

and

$$\frac{1}{\sqrt{a}} \langle f^A, \text{Im } \kappa_{a,0} \rangle = \frac{1}{a} \int_{\mathbb{R}} f^A(x) \text{Im } \kappa \left( \frac{x}{a} \right) dx \rightarrow C_\kappa \Theta, \quad \text{as } a \rightarrow 0. \quad (24)$$

Then,

$$\text{Re} \langle f, \kappa_{a,0} \rangle \in o(\text{Im} \langle f, \kappa_{a,0} \rangle), \quad \text{if } a \rightarrow 0,$$

and therefore, by [Proposition 10](#),

$$\lim_{a \rightarrow 0} \text{sgn} \langle f, \kappa_{a,0} \rangle = i.$$

To establish (23), let  $\epsilon > 0$  be arbitrary. Since  $f^S(0) = 0$  and since  $f^S$  is continuous around the origin, we can find  $\delta > 0$  such that  $|f^S(x)| \leq \epsilon$  for all  $|x| \leq \delta$ . Thus, around the origin we get the estimate

$$\frac{1}{a} \left| \int_{-\delta}^{\delta} f^S(x) \text{Re } \kappa \left( \frac{x}{a} \right) dx \right| \leq \frac{\epsilon}{a} \int_{-\delta}^{\delta} \left| \text{Re } \kappa \left( \frac{x}{a} \right) \right| dx = \epsilon \int_{-\frac{\delta}{a}}^{\frac{\delta}{a}} |\text{Re } \kappa(y)| dy \leq \epsilon \|\text{Re } \kappa\|_{L^1}.$$

Since  $f$  is of polynomial growth and  $\text{Re } \kappa$  is of rapid decay, there exist  $C, M > 0$  and  $N \in \mathbb{N}_0$  such that

$$|f(x)| \leq C|x|^N \quad \text{and} \quad |\text{Re } \kappa(x)| \leq C|x|^{-N-2},$$

for all  $|x| \geq M$ . Hence, there is an  $a_0 > 0$  such that

$$\frac{1}{a} \left| \left( \int_{-\infty}^{-M} + \int_M^{\infty} \right) f^S(x) \text{Re } \kappa \left( \frac{x}{a} \right) dx \right| \leq \frac{C^2}{a} \left| \left( \int_{-\infty}^{-M} + \int_M^{\infty} \right) |x|^N \frac{|a|^{N+2}}{|x|^{N+2}} dx \right| \leq \epsilon,$$

for all  $0 < a < a_0$ . By the rapid decay property of  $\text{Re } \kappa$ , there exists  $0 < a_1 \leq a_0$ , such that

$$\frac{1}{a} \sup_{[\delta, M]} \left| \text{Re } \kappa \left( \frac{x}{a} \right) \right| = \frac{1}{a} \sup_{[a\delta, aM]} |\text{Re } \kappa(x)| \leq \epsilon,$$

for all  $0 < a \leq a_1$ . As  $f$  is locally integrable, it follows that

$$\frac{1}{a} \left| \left( \int_{-M}^{-\delta} + \int_{\delta}^M \right) f^S(x) \text{Re } \kappa \left( \frac{x}{a} \right) dx \right| \leq \frac{1}{a} \int_{-M}^M |f^S(x)| dx \cdot \sup_{[\delta, M]} \left| \text{Re } \kappa \left( \frac{x}{a} \right) \right| \leq C' \epsilon,$$

with a constant  $C'$  independent of  $a$ . Combining all three integral estimates, we have found that for every  $\epsilon > 0$  there exists an  $a_1 > 0$  such that

$$\frac{1}{a} \left| \int_{\mathbb{R}} f^S(x) \text{Re } \kappa \left( \frac{x}{a} \right) dx \right| < (\|\text{Re } \kappa\|_{L^1} + 1 + C') \epsilon, \quad \text{for all } a < a_1,$$

which proves (23).

For (24), we prove the following equivalent statement

$$\frac{1}{a} \left| \int_{\mathbb{R}} f^A(x) \text{Im } \kappa \left( \frac{x}{a} \right) dx - C_\kappa \Theta \right| \rightarrow 0, \quad \text{as } a \rightarrow 0.$$

By the antisymmetry of  $\text{Im } \kappa$ , this can be rewritten as

$$\frac{1}{a} \left| \int_{\mathbb{R}} f^A(x) \text{Im } \kappa \left( \frac{x}{a} \right) dx - \Theta \int_0^{\infty} \text{Im } \kappa(x) dx \right| = \frac{1}{a} \left| \int_{\mathbb{R}} \left( f^A(x) - \text{sgn}(x) \frac{\Theta}{2} \right) \text{Im } \kappa \left( \frac{x}{a} \right) dx \right|.$$

Since  $f^A(x) - \text{sgn}(x) \frac{\Theta}{2}$  is continuous around the origin, (24) is proved similarly to (23).

So far we have assumed that  $\Theta > 0$ . The case  $\Theta < 0$ , is treated analogously, introducing a negative sign. In that case,

$$\lim_{a \rightarrow 0} \text{sgn} \langle f, \kappa_{a,0} \rangle = -i,$$

which completes the proof.  $\square$

By [Theorem 12](#), we can compute the signature of a piecewise polynomial signal, which is discontinuous at the interface.

**Example 13.** The function

$$f(x) = \begin{cases} (x+3)^3, & \text{for } x \geq 0 \\ -(x-1)^2, & \text{for } x < 0 \end{cases}$$

has a jump discontinuity to the right at zero. Thus, by [Theorem 12](#),  $\sigma f(0) = i$ . Away from the origin the function coincides with a polynomial and, by [Corollary 6](#), its signature is therefore equal to zero.

### 2.3. Signature, fractional Laplacians and fractional Hilbert transforms

Next, we investigate the behavior of the signature under the action of fractional powers of the Laplacian and the fractional Hilbert transform. We shall see that the former leaves the signature invariant whereas the latter acts on the signature by a rotation in the complex plane.

We recall that fractional powers of the Laplacian  $(-\Delta)^{\frac{r}{2}}$ , acting on  $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$ , are defined by

$$\widehat{(-\Delta)^{\frac{r}{2}} f} := |\bullet|^r \cdot \widehat{f}, \quad \text{for } r \in \mathbb{R}. \quad (25)$$

We show that the signature is invariant under  $(-\Delta)^{\frac{r}{2}}$ . Again, note that the signature is well defined for  $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$ .

**Theorem 14.** *Let  $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$  and  $r \in \mathbb{R}$ . Then,*

$$\sigma \left( (-\Delta)^{\frac{r}{2}} f \right) (b) = \sigma f(b), \quad \text{for all } b \in \mathbb{R}.$$

*Proof.* Let  $\kappa$  be a signature wavelet. By the translation invariance of the signature, we may assume that  $b = 0$ . Then

$$\begin{aligned} \langle (-\Delta)^{\frac{r}{2}} f, \kappa_{a,0} \rangle &= \left\langle |\bullet|^r \widehat{f}, (\kappa_{a,0})^\vee \right\rangle \\ &= a^{-r} \left\langle \widehat{f}, |a \bullet|^r (\kappa_{a,0})^\vee \right\rangle = a^{-r} \left\langle \widehat{f}, D_{\frac{1}{a}}(|\bullet|^r \kappa^\vee) \right\rangle = a^{-r} \left\langle f, D_a((-\Delta)^{\frac{r}{2}} \kappa) \right\rangle. \end{aligned}$$

Hence, as  $a > 0$ , we obtain

$$\operatorname{sgn} \langle (-\Delta)^{\frac{r}{2}} f, \kappa_{a,0} \rangle = \operatorname{sgn} \langle f, ((-\Delta)^{\frac{r}{2}} \kappa)_{a,0} \rangle.$$

Since the class of signature wavelets is invariant under the application of  $(-\Delta)^{\frac{r}{2}}$ , the result follows.  $\square$

Now we turn to the fractional Hilbert transform, which was first introduced in [\[18\]](#). We follow the definition given in [\[19\]](#). For  $\alpha \in \mathbb{R}$ , the fractional Hilbert transform  $\mathcal{H}^\alpha$  is defined on  $\mathcal{S}'(\mathbb{R})/\mathcal{P}$  by

$$\widehat{\mathcal{H}^\alpha f} := e^{-i\alpha \frac{\pi}{2} \operatorname{sgn}(\bullet)} \cdot \widehat{f}. \quad (26)$$

The following theorem shows that the fractional Hilbert transform  $\mathcal{H}^\alpha$  acts on the signature as multiplication by  $e^{i\alpha \frac{\pi}{2}}$ , i.e., as a rotation in the complex plane.

**Theorem 15.** *Let  $f \in \mathcal{S}'(\mathbb{R})/\mathcal{P}$  and  $b \in \mathbb{R}$ . Then*

$$\sigma(\mathcal{H}^\alpha f)(b) = e^{i\alpha \frac{\pi}{2}} \cdot \sigma f(b). \quad (27)$$

*Proof.* By the translation invariance [\(7\)](#), it is sufficient to prove the result for  $b = 0$ . Let  $a > 0$ . We have

$$\langle \mathcal{H}^\alpha f, \kappa_{a,0} \rangle = \left\langle e^{-i\alpha \frac{\pi}{2} \operatorname{sgn}(\omega)} \widehat{f}, (\kappa_{a,0})^\vee \right\rangle = \int_{\mathbb{R}} e^{-i\alpha \frac{\pi}{2} \operatorname{sgn}(\omega)} \widehat{f}(\omega) (\kappa_{a,0})^\vee(\omega) d\omega.$$

	Sobolev regularity index	Signature
Fractional differentiation	$s_{(-\Delta)^{\frac{r}{2}}f} = s_f - r$	$\sigma((-\Delta)^{\frac{r}{2}}f) = \sigma(f)$
Fractional Hilbert transform	$s_{\mathcal{H}^\alpha f} = s_f$	$\sigma(\mathcal{H}^\alpha f) = e^{i\alpha\frac{\pi}{2}}\sigma(f)$

Table 1: Comparison of the action of differential and convolution operators between the Sobolev regularity index and the signature.

Since  $\text{supp } \kappa_{a,0}^\vee \subset (-\infty, 0]$ , this reduces to

$$\begin{aligned} \langle \mathcal{H}^\alpha f, \kappa_{a,0} \rangle &= \int_{-\infty}^0 e^{-i\alpha\frac{\pi}{2} \text{sgn}(\omega)} \widehat{f}(\omega) (\kappa_{a,0})^\vee(\omega) d\omega \\ &= e^{i\alpha\frac{\pi}{2}} \int_{-\infty}^0 \widehat{f}(\omega) (\kappa_{a,0})^\vee(\omega) d\omega = e^{i\alpha\frac{\pi}{2}} \left\langle \widehat{f}, (\kappa_{a,0})^\vee \right\rangle = e^{i\alpha\frac{\pi}{2}} \langle f, \kappa_{a,0} \rangle. \end{aligned}$$

Thus,

$$\sigma(\mathcal{H}^\alpha f)(0) = \lim_{a \rightarrow 0} \text{sgn} \langle \mathcal{H}^\alpha f, \kappa_{a,0} \rangle = e^{i\alpha\frac{\pi}{2}} \lim_{a \rightarrow 0} \text{sgn} \langle f, \kappa_{a,0} \rangle = e^{i\alpha\frac{\pi}{2}} \sigma f(0),$$

which proves the claim.  $\square$

A particular instant of [Theorem 15](#) is the following result.

**Example 16.** We consider the sign function  $f(x) = \text{sgn}(x)$ . The Hilbert transform is given by  $2 \log |\bullet|$ , regarded in the distributional sense, cf. [\[20, p. 80\]](#). [Theorem 15](#) and [Example 7](#) yield

$$\sigma(\log |\bullet|)(b) = \sigma(\mathcal{H}f)(b) = i \cdot \sigma(f)(b) = \begin{cases} i^2 = -1, & \text{if } b = 0, \\ 0, & \text{else.} \end{cases}$$

Now, let us compare the signature to the local Sobolev regularity. Recall that the *local Sobolev regularity index*  $s_f(b)$  of  $f$  at  $b$  is defined by

$$s_f(b) := \sup \{s \in \mathbb{R} : \exists \varphi \in \mathcal{D}(\mathbb{R}), \varphi(b) \neq 0, \text{ so that } \varphi f \in W^{s,2}(\mathbb{R})\}. \quad (28)$$

(See, for example, [\[13, Chapter 18.1\]](#).) The local Sobolev regularity index is often (implicitly) used in signal analysis since it is characterized by the decay of the moduli of wavelet coefficients, cf. [\[2\]](#). Under the application of the fractional Laplacian, the local Sobolev regularity index of a function decreases by the order of the operator. More precisely, for  $f \in \mathcal{S}'(\mathbb{R})$ ,

$$s_{(-\Delta)^{\frac{r}{2}}f}(b) = s_f(b) - r, \quad (29)$$

see [\[13, Theorem 18.1.31\]](#). On the other hand,  $s_f$  is not altered by the fractional Hilbert transform, that is,

$$s_{\mathcal{H}^\alpha f}(b) = s_f(b). \quad (30)$$

To summarize, the fractional Laplacian changes only the local Sobolev index, whereas the fractional Hilbert transform modifies only the signature; see [Table 1](#). In this sense, the signature can be regarded as being dual to classical local smoothness.

In the examples that we have seen so far, the points of non-zero signature are a subset of the classical singular support. This gives rise to the question whether there is a relation between the classical singular support  $\text{sing supp } f$  and the support of the signature  $\text{supp } \sigma f$ . The following two examples show that, in general, neither one is contained in the other.

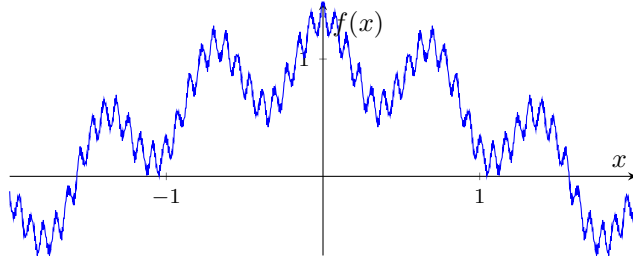


Figure 2: The Weierstrass function for  $r = 0.35$  and  $t = 9$ . Its singular support is the entire real line, but its signature is equal to 0 for every  $b \in \mathbb{R}$ .

**Example 17.** Consider the Weierstraß function (see e.g. [21])

$$f(x) = \sum_{n=0}^{\infty} r^n \cos(t^n x), \quad \text{where } 0 < r < 1 \text{ and } rt \geq 1;$$

see Figure 2. As  $f$  is nowhere differentiable, it follows that  $\text{sing supp } f = \mathbb{R}$ . Now let  $\kappa$  be a signature wavelet such that  $\text{supp } \hat{\kappa} \subset ]1, t[$ . Then, by taking Fourier transforms, we obtain

$$\langle f, \kappa_{a,b} \rangle = \pi \sum_{n=0}^{\infty} \langle \delta_{-t^n} + \delta_{t^n}, E_{-b} D_{\frac{1}{a}} \hat{\kappa} \rangle = \sqrt{a} \pi \sum_{n=0}^{\infty} e^{-ibt^n} \hat{\kappa}(at^n).$$

Now consider the zero sequence  $a_m = t^{-m}$ ,  $m \in \mathbb{N}$ . Since  $\text{supp } \hat{\kappa} \subset ]1, t[$ , we have  $\hat{\kappa}(t^k) = 0$  for any  $k \in \mathbb{Z}$ . Therefore,

$$\langle f, \kappa_{a_m,b} \rangle = t^{-\frac{m}{2}} \pi \sum_{n=0}^{\infty} e^{-ibt^n} \hat{\kappa}(t^{n-m}) = 0.$$

Thus, if  $\text{sgn } \langle f, \kappa_{a,b} \rangle$  converges, then it must necessarily converge to zero. Hence,

$$\sigma f(b) = 0, \quad \text{for all } b \in \mathbb{R}.$$

Therefore, we see that in general

$$\text{sing supp } f \not\subseteq \text{supp } \sigma f. \quad (31)$$

**Example 18.** Let  $f = e^{-\gamma x^2}$  be a Gaussian function with  $\gamma > 0$ , and let  $\kappa$  be any signature wavelet. The singular support of  $f$  is empty because  $f$  is smooth. On the other hand, as the support of  $\hat{\kappa}$  is not empty,

$$\langle f, \kappa_{a,0} \rangle = \langle \hat{f}, (\kappa_{a,0})^\vee \rangle = \sqrt{\pi} \int_{\mathbb{R}} e^{-\frac{\omega^2}{4\gamma}} (\kappa_{a,0})^\vee(\omega) d\omega > 0, \quad \text{for all } a > 0,$$

implying that the signature equals 1 at  $b = 0$ . Thus, in general,

$$\text{supp } \sigma f \not\subseteq \text{sing supp } f. \quad (32)$$

This shows that the converse inclusion of (31) does not hold either.

Example 18 is particularly interesting since signature considers the origin of the Gaussian as a salient point. This seems odd at first sight because the Gaussian function is in  $\mathcal{C}^\infty$ , and thus a very regular function in the classical sense. However, note that the origin is the location of the global symmetry and the global maximum of the Gaussian, and thus may indeed be considered as a salient point. Actually, we see in the next section that signature can be interpreted as an indicator of symmetry or antisymmetry at a salient point.

Signal feature		Local symmetry	Signature
Step to right	$\lrcorner$	antisymmetric	$+i$
Step to left	$\llcorner$	antisymmetric	$-i$
Cusp upwards	$\wedge$	symmetric	$+1$
Cusp downwards	$\vee$	symmetric	$-1$

Table 2: Signatures of frequently occurring feature types. We observe that the signature is imaginary at the antisymmetric and real at the symmetric features.

#### 2.4. Geometric interpretation of the signature

Whereas the local Sobolev regularity index is by definition a measure for the local order of differentiability, the interpretation of signature is not so obvious. We now argue that signature can be considered as an indicator of local symmetry at a singular point.

To this end, let us recall that the signature is imaginary at step discontinuities and real at cusps. This is a first indication that imaginary signatures correspond to locally antisymmetric features and that real signatures indicate locally symmetric features; see Table 2. A more rigorous argument can be derived from the change of the signature under the Hilbert transform. The ordinary Hilbert transform converts the symmetric part of a signal to an antisymmetric function, and vice-versa. Theorem 15 states that the application of the Hilbert transform to a signal induces a multiplication by  $e^{\frac{1}{2}i\pi} = i$  of its signature. Thus, the change of symmetry is directly reflected in the rotation of the signature by  $\frac{\pi}{2}$  in the complex plane. We may explicitly observe this effect for the sign function  $x \mapsto \text{sgn } x$  and its Hilbert transform (in a weak sense), the logarithm  $x \mapsto 2 \log |x|$ . The antisymmetric sign function has a signature of  $i$  at the origin whereas the signature of the logarithm equals  $i \cdot i = -1$ .

Finally, we notice that signature only indicates local symmetry at points with non-zero signature. However, it is not yet clear how these points of non-zero signature can be characterized. This issue will be addressed in a forthcoming paper.

#### 2.5. Randomization of wavelet coefficients

In this section, we show that a signal with randomly perturbed wavelet signs does not contain any salient points in the sense of signature.

Let us introduce random linear operators that are diagonal in a wavelet basis. To this end, let  $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$  be a compactly supported orthonormal wavelet basis of  $L^2(\mathbb{R})$ , where  $j$  stands for the dyadic level and  $k$  for the translation parameter. For  $f \in L^2(\mathbb{R}, \mathbb{R})$ , we have the following decomposition

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \psi_{jk} \rangle \psi_{jk} =: \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} f_{jk} \psi_{jk},$$

where convergence is in  $L^2$ -norm. We define an operator  $A_\varepsilon : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$  of random perturbations of the wavelet signs by

$$A_\varepsilon f := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon_{jk} f_{jk} \psi_{jk}, \quad (33)$$

where  $\{\varepsilon_{jk}\}$  is a Rademacher sequence, i.e. a sequence of independently and identically distributed random variables taking the values  $\pm 1$  with probability  $\frac{1}{2}$ . Regarding the use of Rademacher sequences in the context of random Fourier series, see [22]. More recently, Jaffard [23] considered the use of Rademacher sequences for random perturbations of wavelet series. Our next theorem shows that the signature of  $A_\varepsilon f$  vanishes for almost every realization of the Rademacher sequence.

**Theorem 19.** *Let  $f \in L^2(\mathbb{R})$ ,  $b \in \mathbb{R}$ , and  $A_\varepsilon$  an operator of random sign perturbations as defined in (33). Then*

$$\sigma(A_\varepsilon f)(b) = 0, \text{ for almost every } \varepsilon.$$



*Proof.* We first compute the wavelet coefficients of  $A_\varepsilon f$  relative to a signature wavelet  $\kappa$ . This yields

$$\operatorname{sgn} \langle A_\varepsilon f, \kappa_{2^{-j}, b} \rangle = \operatorname{sgn} \left( \sum_{l,m} \varepsilon_{l,m} \langle f, \psi_{l,m} \rangle \langle \psi_{l,m}, \kappa_{2^{-j}, b} \rangle \right), \quad j > 0. \quad (34)$$

Since the sum in (33) is convergent in the  $\ell^2$ -norm for every realization of  $\{\varepsilon_{lm}\}$ , it follows that

$$(A_\varepsilon f)_{j,b} := \sum_{l,m} \varepsilon_{l,m} \langle f, \psi_{l,m} \rangle \langle \psi_{l,m}, \kappa_{2^{-j}, b} \rangle$$

is a complex-valued bounded random variable. As  $\{\varepsilon_{lm}\}$  is a Rademacher sequence, we have that

$$\operatorname{sgn}(A_{-\varepsilon} f)_{j,b} = -\operatorname{sgn}(A_\varepsilon f)_{j,b}.$$

Let  $I$  be an arc on the unit circle with positive measure. Then we have

$$P(\operatorname{sgn}(A_\varepsilon f)_{j,b} \in I) = P(\operatorname{sgn}(A_\varepsilon f)_{j,b} \in -I),$$

where  $P$  denotes the probability measure associated with the Rademacher sequence  $\{\varepsilon_{lm}\}$ . This implies that for each  $z_0$  on the unit circle

$$P\left(\lim_{j \rightarrow \infty} \operatorname{sgn}(A_\varepsilon f)_{j,b} = z_b\right) = 0.$$

Therefore,

$$\sigma(A_\varepsilon f)(b) = 0, \quad \text{with probability 1.} \quad \square$$

Let us illustrate the above result by considering the characteristic function on  $[0, 1]$ :  $f := \mathbb{1}_{[0,1]}$ . The signal  $f$  has jump discontinuities at 0 and 1, but the shape of the randomly perturbed signal  $A_\varepsilon f$  does not have much in common with the original jump shape of  $f$ . However, since the amplitude of the wavelet coefficients is not altered,  $\langle f, \psi_{jk} \rangle = \langle A_\varepsilon f, \psi_{jk} \rangle$ , it is difficult to distinguish between  $f$  and  $A_\varepsilon f$  using a local signal analysis tool that is based only on the moduli of the wavelet coefficients. Signature, on the other hand, takes into account the random perturbations. The signature of the original function  $f$  is non-zero at 0 and 1, but zero for the perturbed function  $A_\varepsilon f$ .

### 3. Discretization and numerical experiments

In practice, only a finite number of wavelet scales  $\{a_j\}_{j=1}^N$  is available. Furthermore, since we cannot test for convergence in (1) using every signature wavelet, we have to choose a suitable signature wavelet  $\kappa$ . Thus, we have to estimate the signature from the finite set of samples  $\{\operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle\}_{j=1}^N$ .

To motivate our numerical approach, we begin by considering the following elementary convergence result for discrete samples.

**Proposition 20.** *Let  $f$  be a tempered distribution,  $\{a_j\}_{j \in \mathbb{N}}$  a sequence such that  $a_j \rightarrow 0$ , and  $b \in \mathbb{R}$ . If  $\sigma f(b) \neq 0$ , then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle = \sigma f(b) \quad (35)$$

for all signature wavelets  $\kappa$ .

*Proof.* Let  $\kappa$  be a signature wavelet. Since  $\sigma f(b)$  is non-zero, the sequence  $\{\operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle\}_{j \in \mathbb{N}}$  is convergent. We observe that the limit in (35) is the Cesàro sum of this sequence. The claim follows from the fact that, for a convergent sequence, its Cesàro sequence converges to the same limit, cf. [24, Chapter 5.7].  $\square$

Proposition 20 suggests the Cesàro limit (35) as an alternative to computing a non-zero signature  $\sigma f(b)$ . Note that  $\sigma f(b)$  is of modulus 1 and so is the Cesàro limit (35). Furthermore, the elements of the Cesàro sequence

$$\frac{1}{N} \sum_{j=1}^N \operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle, \quad N \in \mathbb{N},$$

are not necessarily of modulus one, but their moduli converge to 1 as  $N$  goes to infinity. This observation motivates the following procedure for the numerical estimation of the signature.

Given a finite number of scale samples  $\{a_j\}_{j=1}^N$ , we interpret the mean of the sequence of discrete signs, given by

$$\bar{w}_b := \frac{1}{N} \sum_{j=1}^N \operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle, \quad (36)$$

as the  $N$ -th element of a Cesàro sequence. If the absolute value  $|\bar{w}_b|$  is close to 1, we consider the Cesàro sequence as being convergent, with  $\operatorname{sgn} \bar{w}_b$  giving an estimate of  $\sigma f(b)$ . On the other hand, a small value of  $|\bar{w}_b|$  suggests a vanishing signature. More precisely, we consider  $|\bar{w}_b|$  to be non zero if it exceeds some empirical threshold parameter  $\tau$  between 0 and 1. Hence, we propose a discrete estimate  $\bar{\sigma} f(b)$  of the signature of the form

$$\bar{\sigma} f(b) := \begin{cases} \operatorname{sgn} \bar{w}_b, & \text{if } |\bar{w}_b| \geq \tau, \\ 0, & \text{elsewhere.} \end{cases} \quad (37)$$

Figure 3 and Figure 4 show numerical experiments based on the above procedure. For all experiments, we used the Meyer-type signature wavelet (6) and scale samples of the form  $a_j = 2^{-\frac{j}{3}}$ , with  $j = 0, 1, \dots, 15$ . The threshold parameter was set to  $\tau = \frac{1}{2}\sqrt{2} \approx 0.7$ . The first experiment (Figure 3) shows that the discrete estimate is in agreement with the theoretical results for pure cusps and pure steps as given in Example 9 and Example 7. In the case of a more complex signal, we also observe that the modulus of the mean  $|\bar{w}_b|$  is large at the salient points; see Figure 4. These experiments show that the procedure proposed above yields a reasonable way to compute the signature numerically.

*Remark 1.* If our sample set  $\{\operatorname{sgn} \langle f, \kappa_{a_j, b} \rangle\}_{j=1}^N$  consists of non-zero elements, then the discrete signature (37) has some interesting connections to directional statistics [26]. In the context of directional statistics, the quantity  $\bar{w}_b$  is called the *mean resultant vector* of the sample set and its sign,  $\operatorname{sgn} \bar{w}_b$ , the *directional mean*. Furthermore, the modulus  $|\bar{w}_b|$  of the mean resultant vector is directly connected to the *directional variance*  $\rho_b$  via

$$\rho_b = 1 - |\bar{w}_b|.$$

These observations allow a statistical interpretation of the discrete signature (37). The sample sequence is considered to be “non-convergent” if the directional variance  $\rho_b$  is large, that is, if the orientations of the samples vary a lot. On the other hand, if the directional variance is small, the directional mean gives an estimate of the signature.

#### 4. Summary and outlook

We have proposed a new tool for signal analysis that is based on complex wavelet signs to discriminate and ultimately classify isolated salient features. We proved that at a jump the signature is imaginary, whereas at a cusp it is real. Furthermore, signature is invariant under fractional Laplacians but serves as a multiplier when acted on by the fractional Hilbert transform. In addition, we showed that singularities regarded in the classical sense need not coincide with singularities described in terms of signature, and that

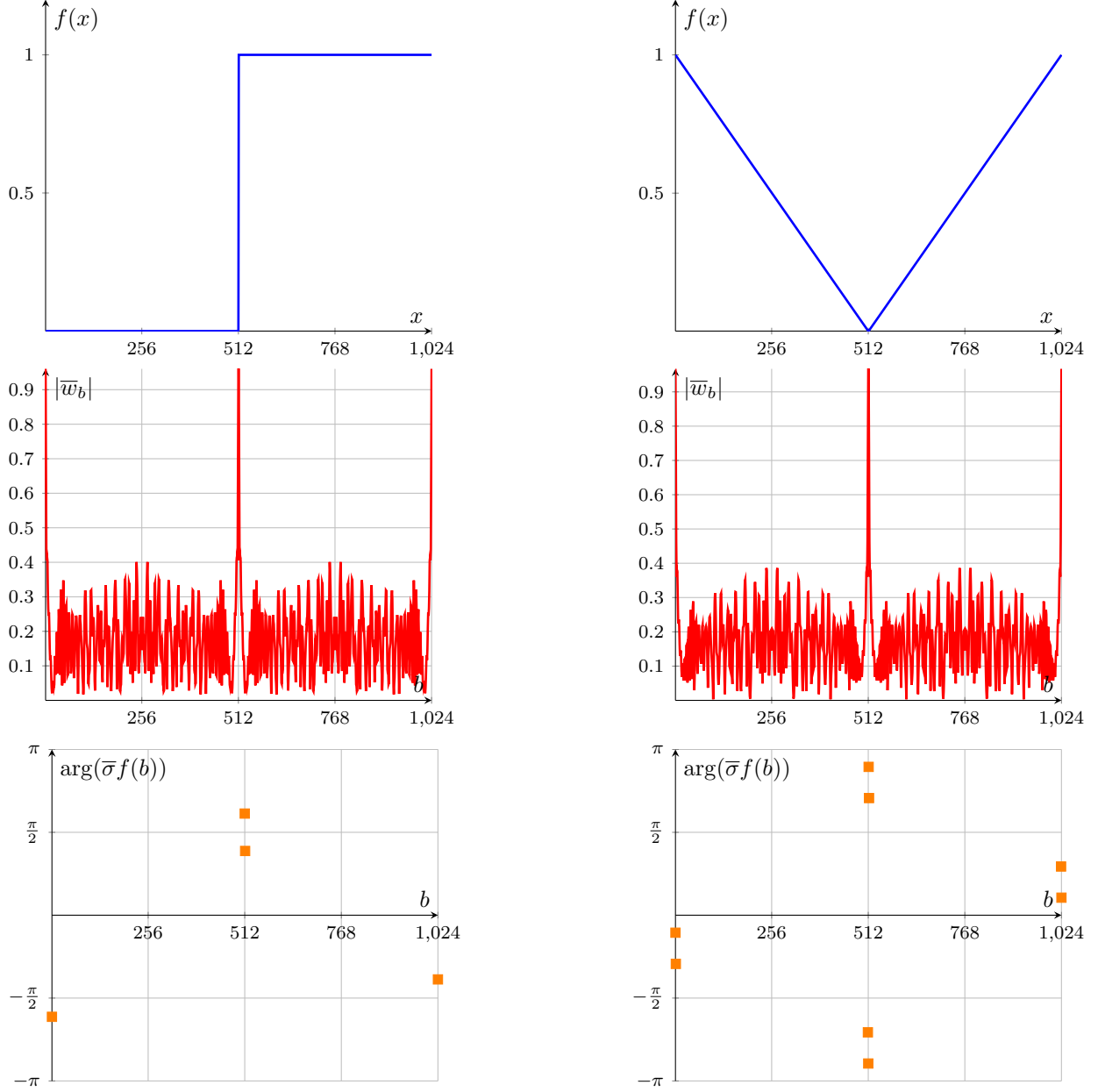


Figure 3: The discrete signature of two sample signals. We observe that the absolute value of the resultant vector  $\overline{w}_b$  is close to one at the salient points, here a step and a cusp, and much lower at the other points (second row). It can also be seen that the peaks are sharply localized, so we do not necessarily require a non-maximum suppression here. In the third row, the discrete signature according to (37) is depicted as a phase angle. We see that at the steps the signature is close to the angles  $\pm \frac{\pi}{2}$ , (right). At the cusp, the signature angles group around  $\pi$ . The experiment validates the theoretical values computed in Example 9 and Example 7. Note that the extra points at the boundaries are due to the periodization of the signal.

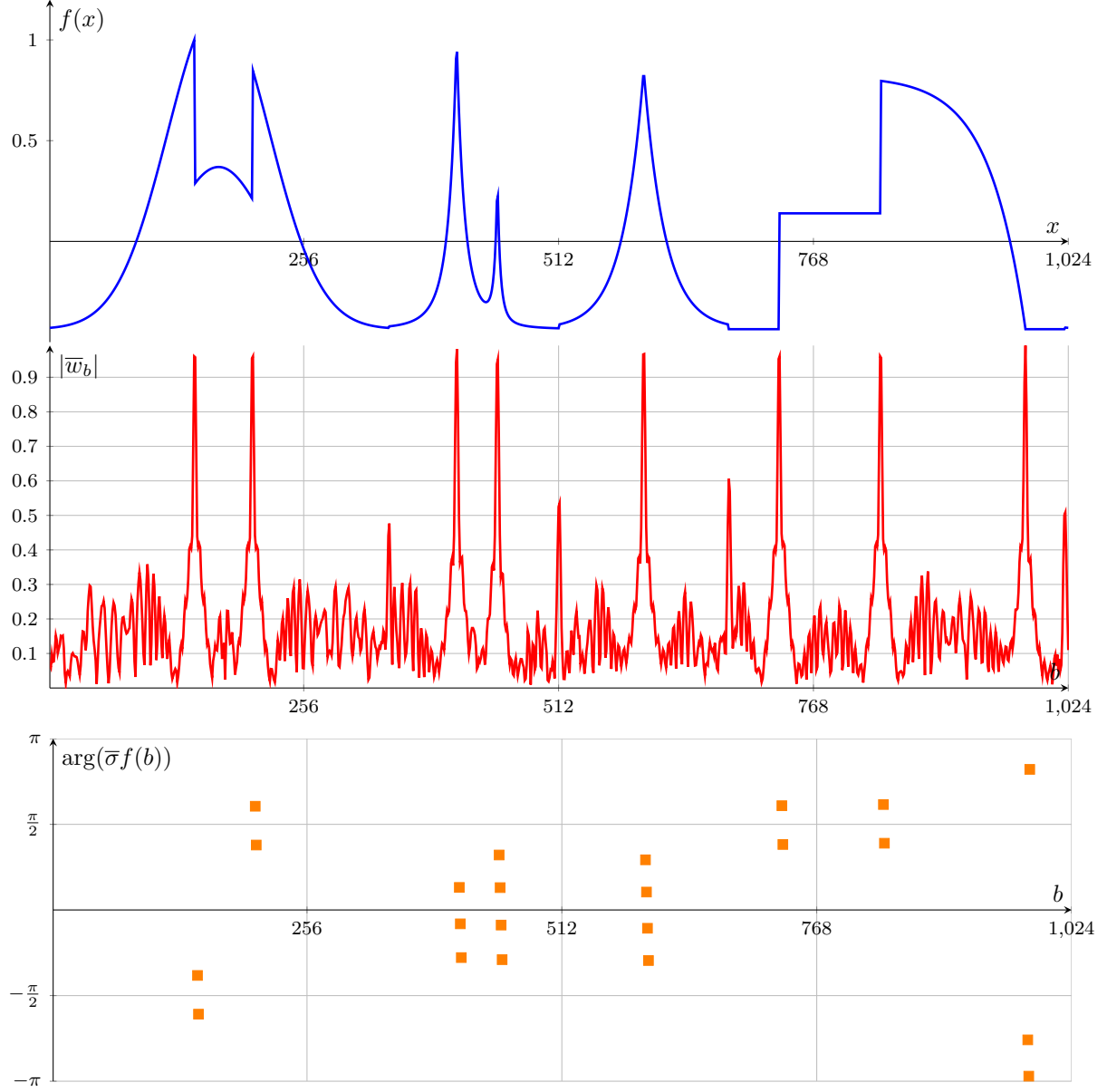


Figure 4: The discrete signature of a sample signal (top) taken from Wavelab [25]. We observe that the absolute value of the mean  $|\overline{w}_b|$  is large at the feature points and much lower at the other points (center). The bottom plot depicts the discrete signature in phase angle representation. We see that the signature clusters around the angles  $\pm\frac{\pi}{2}$  at the step-like points, and around  $\pi$  and  $0$  at cusp-like points. The threshold  $\tau$  is set equal to 0.7 in this experiment. If we choose a lower threshold parameter, say  $\tau = 0.4$ , then the discrete signature would also catch the subtle feature points, like the small jump at  $x = 512$ . However, in that case, we would require a non-maximum suppression to maintain the sharp localization of the pronounced feature points.

signature classifies each point of a signal whose signs are randomly distributed as non-salient. Finally, we proposed a method for the numerical computation and estimation of the signature for real-world signals and used it to validate the theoretical concepts developed in this paper.

Future work includes the analysis of noisy signals, applications to signal reconstruction, and the multi-dimensional extension of the concepts introduced here.

## Appendix A. The Meyer-type signature wavelets

A Meyer-type signature wavelet  $\kappa$  is defined as the inverse Fourier transform of the one-sided window function  $W$  given by

$$W(\omega) := \begin{cases} \cos\left(\frac{\pi}{2}g(5-6\omega)\right), & \text{for } \frac{2}{3} \leq \omega < \frac{5}{6}, \\ 1, & \text{for } \frac{5}{6} \leq \omega < \frac{4}{3}, \\ \cos\left(\frac{\pi}{2}g(3\omega-4)\right), & \text{for } \frac{4}{3} \leq \omega < \frac{5}{3}, \\ 0 & \text{else.} \end{cases} \quad (\text{A.1})$$

Here,  $g$  is a sufficiently smooth function satisfying

$$g(\xi) = \begin{cases} 0, & \text{for } \xi \leq 0, \\ 1 & \text{for } \xi \geq 1, \end{cases} \quad \text{and} \quad g(\xi) + g(1-\xi) = 1, \quad \xi \in \mathbb{R}.$$

For more details, see [27].

## References

- [1] S. Jaffard, Pointwise smoothness, two-microlocalization and wavelet coefficients, *Publicacions Matematiques* 35 (1) (1991) 155–168.
- [2] S. Mallat, A wavelet tour of signal processing: The sparse way. 2009, Academic, Burlington.
- [3] B. Logan Jr, Information in the zero crossings of bandpass signals, *AT & T Technical Journal* 56 (1977) 487–510.
- [4] A. Oppenheim, J. Lim, The importance of phase in signals, *Proceedings of the IEEE* 69 (5) (1981) 529–541.
- [5] M. Morrone, R. Owens, Feature detection from local energy, *Pattern Recognition Letters* 6 (5) (1987) 303–313.
- [6] S. Venkatesh, R. Owens, On the classification of image features, *Pattern Recognition Letters* 11 (5) (1990) 339–349.
- [7] P. Kovess, Image features from phase congruency, *Videre: Journal of Computer Vision Research* 1 (3) (1999) 1–26.
- [8] R. Kronland-Martinet, J. Morlet, A. Grossmann, Analysis of sound patterns through wavelet transforms, *International Journal of Pattern Recognition and Artificial Intelligence* 1 (2) (1987) 273–302.
- [9] A. Grossmann, Wavelet transforms and edge detection, *Stochastic Processes in Physics and Engineering* (1988) 149–157.
- [10] N. Kingsbury, Image processing with complex wavelets, *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 357 (1760) (1999) 2543–2560.
- [11] I. Selesnick, R. Baraniuk, N. Kingsbury, The dual-tree complex wavelet transform, *Signal Processing Magazine, IEEE* 22 (6) (2005) 123–151.
- [12] M. Holschneider, *Wavelets – An Analysis Tool*, Oxford Science Publications, 1995.
- [13] L. Hörmander, The analysis of linear partial differential operators III: Pseudo-differential operators, Vol. 3, Springer verlag, 2007.
- [14] S. Jaffard, Beyond Besov spaces part 2: Oscillation spaces, *Constructive Approximation* 21 (2005) 29–61.
- [15] M. Storath, Amplitude and sign decompositions by complex wavelets, Ph.D. thesis, Technische Universität München, Germany (to appear at the end of 2012).
- [16] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, 2nd Edition, Springer Verlag, 1990.
- [17] I. Gel'fand, G. Shilov, E. Saletan, *Generalized functions*, Vol. 1, Academic Press New York, 1964.
- [18] A. Lohmann, D. Mendlovic, Z. Zalevsky, Fractional hilbert transform, *Optics letters* 21 (4) (1996) 281–283.
- [19] Y. Luchko, H. Matérn, J. Trujillo, Fractional fourier transform and some of its applications, *Fract. Calc. Appl. Anal.* 11 (4) (2008) 457–470.
- [20] Y. Meyer, D. H. Salinger, *Wavelets and Operators*, Cambridge University Press, 1992.  
URL <http://dx.doi.org/10.1017/CB09780511623820>
- [21] G. Hardy, Weierstrass's non-differentiable function, *Trans. Amer. Math. Soc* 17 (3) (1916) 301–325.
- [22] J.-P. Kahane, *Some Random Series of Functions*, 2nd Edition, Cambridge University Press, 1985.
- [23] S. Jaffard, Beyond Besov spaces part 1: Distributions of wavelet coefficients, *Journal of Fourier Analysis and Applications* 10 (2004) 221–246.
- [24] G. Hardy, *Divergent series*, Oxford University Press, 1949.
- [25] D. Donoho, A. Maleki, M. Shahrnam, *Wavelab 850*, Software toolkit for time-frequency analysis.
- [26] N. Fisher, *Statistical analysis of circular data*, Cambridge Univ Pr, 1996.
- [27] I. Daubechies, *Ten lectures on wavelets*, Vol. 61, Society for Applied and Industrial Mathematics, 1992.